Unlocking the Standard Model. II. 1 generation of quarks. Masses and couplings.

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UNLOCKING THE STANDARD MODEL

II . 1 GENERATION OF QUARKS . MASSES AND COUPLINGS

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Abstract: We continue investigating the Standard Model for one generation of fermions and two parity-transformed Higgs doublets K and H advocated for in a previous work [1], using the one-to-one correspondence, demonstrated there, between their components and bilinear quark operators. We show that all masses and couplings, in particular those of the two Higgs bosons ς and ξ, are determined by low energy considerations. The mass of the "quasi-standard" Higgs boson, ξ, is $m_\xi \approx \sqrt{2}\, m_\pi$; it is coupled to u and d quarks with identical strengths. The mass of the lightest one, ς, is $m_\varsigma \approx m_\pi \dfrac{f_\pi}{\sqrt{2} m_W/g} \approx 68\, KeV$; it is very weakly coupled to matter except hadronic matter. The ratio of the two Higgs masses is that of the two scales involved in the problem, the weak scale $\sigma = \frac{2m_W}{g} \approx 250\, GeV$ and the chiral scale $v = f_\pi$, which are also, up to a factor $1/\sqrt{2}$, the respective vacuum expectation values of the two Higgs bosons. They can freely coexist and be accounted for. The dependence of m_ς and m_ξ on m_π, that is, on quark masses, suggests their evolution when more generations are added. Fermions get their masses from both Higgs multiplets. The theory definitely stays in the perturbative regime.

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1 Introduction

The extension of the Standard Model [2] for one generation of fermions advocated for in [1] is endowed with two Higgs doublets, a "chiral" doublet

$$K = \left(\begin{array}{c} \mathfrak{p}^1 - i\mathfrak{p}^2 \\ -(\mathfrak{s}^0 + \mathfrak{p}^3) \end{array}\right), \quad <\mathfrak{s}^0>= \frac{v}{\sqrt{2}}, \tag{1}$$

and a "weak" doublet

$$H = \left(\begin{array}{c} \mathfrak{s}^1 - i\mathfrak{s}^2 \\ -(\mathfrak{p}^0 + \mathfrak{s}^3) \end{array}\right), \quad <\mathfrak{s}^3>= \frac{\sigma}{\sqrt{2}}, \tag{2}$$

both isomorphic to the Higgs doublet of the Standard Model [2]. It constitutes the "smallest maximal extension" of the Glashow-Salam-Weinberg model. It is maximal in the sense that it incorporates all possible $J=0$ scalars

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(and pseudoscalars) that are expected for a given number of generations, and it is the smallest extension because it does not invoke *a priori* any physics "beyond the Standard Model" now any new type of particle.

$\mathfrak{s}^0$ and $\mathfrak{s}^3$ have non-vanishing vacuum expectation values (VEV's) as written in (2). In there, the symbols "$\mathfrak{s}$" and "$\mathfrak{p}$" stand respectively for "scalar" and "pseudoscalar", such that H and K are parity transformed of each other. Their components that we call generically h^0, h^1, h^2, h^3 transform respectively by $SU(2)_L$ and $SU(2)_R$ according to

$$\begin{aligned} T^i_L \,.\, h^j &= -\tfrac{1}{2}\left(i\,\epsilon_{ijk}h^k + \delta_{ij}\,h^0\right), \\ T^i_L \,.\, h^0 &= -\tfrac{1}{2}\,h^i, \end{aligned} \tag{3}$$

and

$$\begin{aligned} T^i_R \,.\, h^j &= -\tfrac{1}{2}\left(i\,\epsilon_{ijk}h^k - \delta_{ij}\,h^0\right), \\ T^i_R \,.\, h^0 &= +\tfrac{1}{2}\,h^i. \end{aligned} \tag{4}$$

The main steps of this works are the following. In section 2 we give the general formula for the mass of the $\vec{W}$ gauge bosons in terms of the two VEV's $<\mathfrak{s}^0>$ and $<\mathfrak{s}^3>$. In section 3 we introduce Yukawa couplings of quarks to both Higgs doublets K and H. It could have looked more natural to first introduce the scalar potential, but it turns out that the latter gets strongly constrained by the former. After giving their general expression, from which we extract the u and d quark masses in terms of $<\mathfrak{s}^0>$ and $<\mathfrak{s}^3>$, we investigate in section 4 their low energy limit by using the one-to-one correspondence demonstrated in [1] between K, H and 4-sets of bilinear quark operators. At this limit, renormalizability is not a concern and Yukawa couplings can be rewritten in a very simple form in which, in particular, symmetries clearly show up. Using the Partially Conserved Axial Current hypothesis (PCAC) [3] [4] [5] and the Gell-Mann-Oakes-Renner (GMOR) [6] [5] relation enables to account for the mass of the pions and to determine the values of all but one Yukawa parameters. The last one is obtained by identifying the Goldstones of the spontaneously broken weak $SU(2)_L$ symmetry. A last constraint results from considering the $\pi^0 - \eta$ system and requesting that it be devoid of any tachyonic state. This determines the quantity $(m_u - m_d) < \bar{u}u - \bar{d}d >$ ($(m_u + m_d) < \bar{u}u + \bar{d}d >$) is determined by the GMOR relation). We comment at length on fermion masses, and the important role of both Higgs doublets in their generation. After gathering the values of the parameters in section 5, section 6 is devoted to the scalar potential. $V(K, H)$ is chosen to be invariant by the chiral group $U(2)_L \times U(2)_R$, which clearly identifies the Goldstone of chiral symmetry breaking. It only depends on two parameters, one quadratic and one quartic coupling. At low energy, it receives corrections from the bosonised (low energy) form of the Yukawa couplings, which yields an effective potential $V_{eff}(K, H)$. A last constraint comes from minimizing V_{eff} at the known VEV's of the two Higgs bosons, which reproduce the pion and W masses. It determines the value of the quartic coupling and the masses of the two Higgs bosons. In section 7, we determine their couplings to quarks, gauge bosons and leptons. Section 8 provides some additional considerations concerning symmetries, Goldstone and pseudo-Goldstone bosons. Several symmetries are at work and some fields play dual roles. We focus in particular on the custodial $SU(2)$ symmetry and on the respective roles of $< \bar{u}u + \bar{d}d >$ and $< \bar{u}u - \bar{d}d >$. Section 9 gives some remarks concerning more generations. Section 10 is a brief conclusion.

2 Kinetic terms for the Higgs doublets and gauge boson masses

The masses of gauge bosons arise from the kinetic terms

$$\Big((D_\mu K)^\dagger D^\mu K + (D_\mu H)^\dagger D^\mu H\Big) \tag{5}$$

for the two Higgs doublets K and H. D_μ is the covariant derivative with respect to the group $SU(2)_L$ of weak interactions. Owing to the laws of transformations (3), the VEV's of $\mathfrak{s}^0$ and $\mathfrak{s}^3$ generate a mass m_W for the $\vec{W}$ gauge bosons

$$m_W^2 = \frac{g^2}{2}\left(<\mathfrak{s}^0>^2 + <\mathfrak{s}^3>^2 \right) = g^2\,\frac{v^2+\sigma^2}{4}, \tag{6}$$

in which g is the $SU(2)_L$ coupling constant

$$g \approx .61\ . \tag{7}$$

3 Yukawa couplings

We choose to first introduce Yukawa couplings because their low-energy limit (see section 4) will in particular constrain the effective scalar potential.

3.1 General expression

Quarks must be coupled to the two Higgs doublets K and H. Introducing the couplings ρ_u and ρ_d to K and λ_u and λ_d to H, the Yukawa Lagrangian writes [1]

$$\begin{aligned}\mathcal{L}_{Yukawa} = &+ \rho_d \left(\overline{u_L}\,\overline{d_L} \right) K\, d_R - \rho_u \left(\overline{u_L}\,\overline{d_L} \right) (i\tau^2 K^*)\, u_R \\ &+ \lambda_d \left(\overline{u_L}\,\overline{d_L} \right) H\, d_R + \lambda_u \left(\overline{u_L}\,\overline{d_L} \right) (i\tau^2 H^*)\, u_R \\ &+ h.c.,\end{aligned} \tag{8}$$

which gives, explicitly,

$$\begin{aligned}\mathcal{L}_{Yukawa} = &- \left[\delta_1 \frac{v}{\sqrt{2}\mu^3}(\bar u u + \bar d d) + \kappa_{12}\frac{\sigma}{\sqrt{2}\nu^3}(\bar u u - \bar d d)\right]\mathfrak{s}^0 - \left[\delta_{12}\frac{v}{\sqrt{2}\mu^3}(\bar u u + \bar d d) + \delta_2 \frac{\sigma}{\sqrt{2}\nu^3}(\bar u u - \bar d d)\right]\mathfrak{s}^3 \\ &+ \left[\delta_1 \frac{v}{\sqrt{2}\mu^3}\Big(\bar u\gamma_5 d\,\mathfrak{p}^- + \bar d\gamma_5 u\,\mathfrak{p}^+ + (\bar u\gamma_5 u - \bar d\gamma_5 d)\,\mathfrak{p}^3\Big) + \kappa_{12}\frac{\sigma}{\sqrt{2}\nu^3}\Big(\bar d u\,\mathfrak{p}^+ - \bar u d\,\mathfrak{p}^- + (\bar u\gamma_5 u + \bar d\gamma_5 d)\,\mathfrak{p}^3\Big)\right] \\ &- \left[\delta_{12}\frac{v}{\sqrt{2}\mu^3}\Big(\bar d\gamma_5 u\,\mathfrak{s}^+ - \bar u\gamma_5 d\,\mathfrak{s}^- - (\bar u\gamma_5 u - \bar d\gamma_5 d)\,\mathfrak{p}^0\Big) + \delta_2\frac{\sigma}{\sqrt{2}\nu^3}\Big(\bar d u\,\mathfrak{s}^+ + \bar u d\,\mathfrak{s}^- - (\bar u\gamma_5 u + \bar d\gamma_5 d)\,\mathfrak{p}^0\Big)\right].\end{aligned} \tag{9}$$

In (8) and (9) the signs have been set such that for positive $<\mathfrak{s}^0>$ and $<\mathfrak{s}^3>$, the fermion masses are positive for positive $\rho_{u,d}$ and $\lambda_{u,d}$ (given that a fermion mass term is of the form $-m\bar\psi\psi$). We introduced in (9) the parameters with dimension $[mass]^2$

$$\begin{aligned}\delta_1 &= \frac{\rho_u+\rho_d}{2}\,\frac{\sqrt{2}\mu^3}{v},\\ \kappa_{12} &= \frac{\rho_u-\rho_d}{2}\,\frac{\sqrt{2}\nu^3}{\sigma},\\ \delta_{12} &= \frac{\lambda_u+\lambda_d}{2}\,\frac{\sqrt{2}\mu^3}{v},\\ \delta_2 &= \frac{\lambda_u-\lambda_d}{2}\,\frac{\sqrt{2}\nu^3}{\sigma}.\end{aligned} \tag{10}$$

[1] τ^2 is the Pauli matrix $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. The doublets $\tilde K \equiv i\tau^2 K^*$ and $\tilde H \equiv i\tau^2 H^*$ are isomorphic to K and H.

3.2 Fermion masses

We define the two quantum Higgs fields ς and ξ by shifting the scalar fields $\mathfrak{s}^0$ and $\mathfrak{s}^3$ occurring respectively in the Higgs doublets K and H (see (1),(2)) according to

$$\mathfrak{s}^0 =< \mathfrak{s}^0 > +\varsigma, \quad \mathfrak{s}^3 =< \mathfrak{s}^3 > +\xi. \tag{11}$$

The two VEV's (given in (1) and (2)) contribute to the fermion masses according to

$$m_u = \rho_u < \mathfrak{s}^0 > +\lambda_u < \mathfrak{s}^3 >= \frac{v\rho_u + \sigma\lambda_u}{\sqrt{2}}, \quad m_d = \rho_d < \mathfrak{s}^0 > +\lambda_d < \mathfrak{s}^3 >= \frac{v\rho_d + \sigma\lambda_d}{\sqrt{2}}. \tag{12}$$

Additional remarks concerning fermion masses are written in subsection 4.4.

4 The low energy limit

At low energy we use the one-to-one correspondence between K, H and

$$\begin{aligned}
\mathfrak{K} = \frac{1}{\sqrt{2}}\frac{v}{\mu^3}\begin{pmatrix} \phi^1 - i\phi^2 \\ -(\phi^0 + \phi^3)\end{pmatrix} &= \frac{v\sqrt{2}}{\mu^3}\begin{pmatrix} \bar d\gamma_5 u \\ -\frac{1}{2}(\bar u u + \bar d d) - \frac{1}{2}(\bar u\gamma_5 u - \bar d\gamma_5 d)\end{pmatrix} \equiv \begin{pmatrix} \mathfrak{k}^1 - i\mathfrak{k}^2 \\ -(\mathfrak{k}^0 + \mathfrak{k}^3)\end{pmatrix}, \\
< \bar u u + \bar d d > &= \mu^3, \\
\mathfrak{H} = \frac{1}{\sqrt{2}}\frac{\sigma}{\nu^3}\begin{pmatrix} \xi^1 - i\xi^2 \\ -(\xi^0 + \xi^3)\end{pmatrix} &= \frac{\sigma\sqrt{2}}{\nu^3}\begin{pmatrix} \bar d u \\ -\frac{1}{2}(\bar u\gamma_5 u + \bar d\gamma_5 d) - \frac{1}{2}(\bar u u - \bar d d)\end{pmatrix} \equiv \begin{pmatrix} \mathfrak{h}^1 - i\mathfrak{h}^2 \\ -(\mathfrak{h}^0 + \mathfrak{h}^3)\end{pmatrix}, \\
< \bar u u - \bar d d > &= \nu^3,
\end{aligned} \tag{13}$$

that has been established in [1] and identify accordingly

$$(\mathfrak{s}^0, \mathfrak{p}^1, \mathfrak{p}^2, \mathfrak{p}^3) \simeq (\mathfrak{k}^0, \mathfrak{k}^1, \mathfrak{k}^2, \mathfrak{k}^3), \quad (\mathfrak{p}^0, \mathfrak{s}^1, \mathfrak{s}^2, \mathfrak{s}^3) \simeq (\mathfrak{h}^0, \mathfrak{h}^1, \mathfrak{h}^2, \mathfrak{h}^3). \tag{14}$$

4.1 Rewriting Yukawa couplings

The first consequence of this correspondence is that, defining

$$m_{12}^2 = \kappa_{12} + \delta_{12}, \tag{15}$$

and expressing the bilinear quark operators in (9) in terms of the components $(\mathfrak{s}^0, \vec{\mathfrak{p}}), (\mathfrak{p}^0, \vec{\mathfrak{s}})$ of K and H, the Yukawa couplings (9) rewrite

$$\mathcal{L}^{eff}_{Yukawa} = -\delta_1\, K^\dagger K - \frac{1}{2}m_{12}^2\,(K^\dagger H + H^\dagger K) - \delta_2\, H^\dagger H, \tag{16}$$

or, indifferently, since renormalizability is not an issue at low energy, as a sum of 4-fermion interactions

$$\mathcal{L}^{eff}_{Yukawa} = -\delta_1\, \mathfrak{K}^\dagger \mathfrak{K} - \frac{1}{2}m_{12}^2\,(\mathfrak{K}^\dagger \mathfrak{H} + \mathfrak{H}^\dagger \mathfrak{K}) - \delta_2\, \mathfrak{H}^\dagger \mathfrak{H}. \tag{17}$$

This bosonised form of the Yukawa couplings, only valid at low energy, will be later added to the scalar potential $V(K, H)$ to define the low energy effective potential $V_{eff}(K, H)$ (see subsection 6.2).

4.2 PCAC and the Gell-Mann-Oakes-Renner relation

Kinetic terms together with Yukawa couplings include in particular

$$(\partial_\mu K)^\dagger \partial^\mu K - \delta_1 K^\dagger K - \frac{1}{2} m_{12}^2 \,(K^\dagger H + H^\dagger K) + (\partial_\mu H)^\dagger \partial^\mu H - \delta_2 \, H^\dagger H + \ldots \tag{18}$$

and we now raise the issue whether, at low energy, the charged components of K can be identified with the charged pions. As we shall see in subsection 4.3 below, both δ_2 and m_{12}^2 have to vanish: the first to ensure that the breaking of the weak $SU(2)_L$ is accompanied by three true Goldstone bosons, and the second to ensure that the $\mathfrak{p}^0 - \mathfrak{p}^3$ system does not exhibit a tachyonic state. Eq. (18) reduces then to standard kinetic terms for unmixed doublets. Furthermore, the scalar potential will be chosen in such a way that the three pseudoscalar bosons inside K are Goldstone bosons in the absence of Yukawa couplings. So, due to the mass term proportional to δ_1, the three "pions" inside K get a mass m at the simple condition that $\delta_1 = \frac{1}{2} m^2$.

Owing to the Partially Conserved Axial Current (PCAC) hypothesis [3][4]

$$i(m_u + m_d)\, \bar u \gamma_5 d = \sqrt{2} f_\pi m_\pi^2 \, \pi^+, \tag{19}$$

which identifies the interpolating pion field with a bilinear quark operator, and to the corresponding Gell-Mann-Oakes-Renner relation [6]

$$(m_u + m_d) < \bar u u + \bar d d >= 2 f_\pi^2 m_\pi^2, \tag{20}$$

$\mathfrak{p}^+ \equiv \mathfrak{p}^1 + i \mathfrak{p}^2 = \frac{v\sqrt{2}}{\mu^3} \bar d \gamma_5 u$ as it is defined in (16) and (14) can be identified at low energy with

$$\mathfrak{p}^\pm \simeq -\frac{iv}{f_\pi} \pi^\pm. \tag{21}$$

So, the kinetic terms $(\partial_\mu K)^\dagger \partial^\mu K$, which contain in particular $\partial_\mu \mathfrak{p}^+ \partial^\mu \mathfrak{p}^- \equiv \big(\partial_\mu \mathfrak{p}^1 \partial^\mu \mathfrak{p}^1 + \partial_\mu \mathfrak{p}^2 \partial^\mu \mathfrak{p}^2\big)$, will be normalized in the standard way if

$$v = f_\pi, \tag{22}$$

such that

$$\mathfrak{p}^\pm \simeq -i \pi^\pm. \tag{23}$$

Then, the term proportional to δ_1 in (18) is a suitable pion mass terms if

$$\delta_1 = m_\pi^2. \tag{24}$$

Going back to the definition of δ_1 in (10) and using (22) and (20), (24) corresponds to

$$\rho_u + \rho_d = \frac{m_u + m_d}{\sqrt{2} f_\pi}. \tag{25}$$

Since $f_\pi \ll m_W$, (22) plugged into (6) entails

$$\sigma \approx \frac{2 m_W}{g}, \tag{26}$$

which shows that the $\vec W$'s get their mass essentially from the VEV of $\mathfrak{s}^3$. The ratio of the VEV's of the two Higgs doublets comes out accordingly as

$$\tan\beta = \frac{< \mathfrak{s}^3 >}{< \mathfrak{s}^0 >} = \frac{\sigma/\sqrt{2}}{v/\sqrt{2}} \approx \frac{2 m_W}{g f_\pi} \approx 2856. \tag{27}$$

They correspond respectively to the weak (m_W) and chiral (f_π) scale. Both scales can now coexist, unlike in the genuine Glashow-Salam-Weinberg model where the parity-transformed H of the Higgs doublet K is missing.

Eqs. (12) and (26) then determine λ_u and λ_d

$$(\lambda_u+\lambda_d)=\frac{g}{2\sqrt{2}m_W}(m_u+m_d),\quad (\lambda_u-\lambda_d)=\frac{g}{\sqrt{2}m_W}\left((m_u-m_d)-\frac{f_\pi}{\sqrt{2}}(\rho_u-\rho_d)\right), \tag{28}$$

that is

$$\lambda_u=g\,\frac{3m_u-m_d-2\sqrt{2}f_\pi(\rho_u-\rho_d)}{4\sqrt{2}\,m_W},\quad \lambda_d=g\,\frac{3m_d-m_u+2\sqrt{2}f_\pi(\rho_u-\rho_d)}{4\sqrt{2}\,m_W}, \tag{29}$$

in terms of $\rho_u-\rho_d$ which is, at this point, still undetermined.

4.3 Goldstones and pseudo-Goldstones

4.3.1 The charged Goldstones of the broken $SU(2)_L$

Since $<\mathfrak{s}^3>$ provides most of the mass of the $\vec{W}$'s, the charged Goldstone bosons of the broken $SU(2)_L$ weak symmetry are, to a very good approximation, the excitations of $\mathfrak{s}^3$ by the generators T_L^+ and T_L^-, that is $\mathfrak{s}^+$ and $\mathfrak{s}^-\in H$.

However, the $SU(2)_L$ invariant Yukawa couplings that need to be introduced to provide fermions with "soft" masses also give, at low energy, among other couplings, a "soft" mass to $\mathfrak{s}^+$ and $\mathfrak{s}^-$ through the term proportional to δ_2. The situation for $\mathfrak{s}^+$ and $\mathfrak{s}^-$ is different from that of the pions which can become pseudo-Goldstone bosons and stay as physical particles. The spontaneously broken $SU(2)_L$ symmetry requires true Goldstones, which can only go along with

$$\delta_2=0, \tag{30}$$

which is accordingly a side-effect of weak symmetry breaking. Looking at (10), one could think that $\nu^3\equiv<\bar{u}u-\bar{d}d>=0$ could be a solution to $\delta_2=0$. However, we shall see later in subsection 8.1 that $<\bar{u}u>$ must be different from $<\bar{d}d>$ as a trigger of both weak and custodial symmetry breaking. So, (30) entails

$$\lambda_u=\lambda_d=\frac{g}{4\sqrt{2}\,m_W}(m_u+m_d). \tag{31}$$

By (28), (31) determines

$$\rho_u-\rho_d=\frac{\sqrt{2}(m_u-m_d)}{f_\pi}, \tag{32}$$

and, combined with (25),

$$\rho_u=\frac{3m_u-m_d}{2\sqrt{2}f_\pi},\quad \rho_d=\frac{3m_d-m_u}{2\sqrt{2}f_\pi}. \tag{33}$$

4.3.2 The $\mathfrak{p}^3-\mathfrak{p}^0$ system

The $(\mathfrak{p}^3,\mathfrak{p}^0)$ or $(\mathfrak{k}^3,\mathfrak{h}^0)$ or, equivalently (π^0,η) system gets endowed by the Yukawa couplings with a mass matrix

$$\frac{1}{2}\begin{pmatrix}2\delta_1 & m_{12}^2\\ m_{12}^2 & 2\delta_2\end{pmatrix}. \tag{34}$$

However, since δ_2 has been fixed to zero in subsection 4.3.1, this system now exhibits a tachyonic state unless

$$m_{12}^2=0\Leftrightarrow(\rho_u-\rho_d)\frac{\nu^3}{\sigma}=-(\lambda_u+\lambda_d)\frac{\mu^3}{v}\Leftrightarrow\frac{m_u-m_d}{m_u+m_d}=-\frac{1}{2}\frac{\mu^3}{\nu^3}\equiv-\frac{1}{2}\frac{<\bar{u}u+\bar{d}d>}{<\bar{u}u-\bar{d}d>}, \tag{35}$$

in which we have used (15).(10), (31), (33) and the definitions of μ^3 and ν^3 that were introduced in (13).

Eq. (35) is equivalent to [2]

$$\frac{<\bar{d}d>}{<\bar{u}u>} = \frac{3m_u - m_d}{m_u - 3m_d}. \tag{36}$$

When this is realized, $\mathfrak{p}^0$ is a true Goldstone and $\mathfrak{p}^3$ keeps its mass m_π^2. They do not mix. This fits the picture of $\mathfrak{p}^0$ being the third Goldstone boson of the broken $SU(2)_L$ symmetry, and $\mathfrak{p}^3$ being the neutral member of the triplet of pseudo-Goldstone bosons of the broken chiral symmetry $SU(2)_L \times SU(2)_R$ down to the diagonal $SU(2)$. Other considerations concerning symmetries will be given in section 8.

4.3.3 No scalar-pseudoscalar coupling

Yukawa couplings are seen on (9) to potentially generate couplings between charged scalars, for example $\mathfrak{s}^- = \frac{\sigma}{\sqrt{2}\nu^3}\bar{d}u$ and pseudoscalars, for example $\mathfrak{p}^+$. It is the second important effect of the condition $m_{12}^2 \equiv \delta_{12}+\kappa_{12} = 0$ obtained in subsection 4.3.2 to cancel these transitions.

4.3.4 The unitary gauge. Leptonic decays of pions

In the unitary gauge the crossed couplings between the $\vec{W}$ gauge bosons and the (derivative of the) $SU(2)_L$ Goldstone bosons $\mathfrak{p}^0, \mathfrak{s}^+, \mathfrak{s}^-$ are canceled, which leaves untouched the similar couplings between $\vec{W}$ and the three pions. Their proportionality to $v = f_\pi$ yields in particular leptonic decays of pions in agreement with the standard PCAC calculation.

4.4 Fermion masses versus the low energy effective Lagrangian

Fermions receive their masses from the VEV's of the two Higgs doublets K and H. From (12) and the values of the parameters that have been determined (see also section 5 below), it appears that $<\mathfrak{s}^3> \in H$ contribute to the u and d masses by the same amount $\frac{\sigma\lambda_u}{\sqrt{2}} = \frac{\sigma\lambda_d}{\sqrt{2}} = \frac{m_u+m_d}{4}$. Then, $<\mathfrak{s}^0> \in K$ contributes to the u mass by $\frac{v\rho_u}{\sqrt{2}} = \frac{3m_u - m_d}{4}$ and to the d mass by $\frac{v\rho_d}{\sqrt{2}} = \frac{3m_d - m_u}{4}$.

The second point is the inadequacy to calculate quark masses from the low energy effective expression (16) of the Yukawa couplings and its set of parameters determined by low energy considerations. When plugged into (16) the conditions $\delta_2 = 0$ and $m_{12}^2 \equiv \delta_{12} + \kappa_{12} = 0$ demonstrated respectively in (30) and in (35) entail that quark masses come from the sole Higgs doublet K, by $-\delta_1 K^\dagger K$. Going back to quark fields and writing it for example as the product $-\delta_1 K^\dagger \mathfrak{K}$ of scalar fields K times their equivalents in terms of bilinear quark operators $\mathfrak{K}$, which respects renormalizability, $\mathcal{L}_{Yukawa}^{eff}$ does, through quark-antiquark condensation, generate quark masses. They however come out as $-\delta_1 \frac{v^2}{2\mu^3}(\bar{u}u + \bar{d}d) = -\frac{m_u+m_d}{4}(\bar{u}u+\bar{d}d)$, which is different from the masses obtained from the original Lagrangian (9)

$$-\delta_1 \frac{v}{\sqrt{2}\mu^3}(\bar{u}u+\bar{d}d) <\mathfrak{s}^0> + \delta_{12}\left(\frac{\sigma}{\sqrt{2}\nu^3}(\bar{u}u-\bar{d}d) <\mathfrak{s}^0> - \frac{v}{\sqrt{2}\mu^3}(\bar{u}u+\bar{d}d) <\mathfrak{s}^3>\right); \tag{37}$$

using the expression for δ_{12} deduced from (10) and (28), the genuine Lagrangian (37) yields the mass terms

$$\begin{aligned} -\delta_1\frac{v^2}{2\mu^3}(\bar{u}u+\bar{d}d) + \delta_{12}\frac{v\sigma}{2}\left(\frac{\bar{u}u-\bar{d}d}{\nu^3} - \frac{\bar{u}u+\bar{d}d}{\mu^3}\right) &\stackrel{(39)}{=} -\frac{f_\pi^2 m_\pi^2}{2}\frac{\bar{u}u+\bar{d}d}{\mu^3} + \frac{f_\pi^2 m_\pi^2}{2}\left(\frac{\bar{u}u-\bar{d}d}{\nu^3} - \frac{\bar{u}u+\bar{d}d}{\mu^3}\right) \\ &\stackrel{(39)}{=} \underbrace{-\frac{1}{4}(m_u+m_d)(\bar{u}u+\bar{d}d)}_{\text{from }\delta_1} \underbrace{-\frac{1}{2}(m_u-m_d)(\bar{u}u-\bar{d}d)}_{\text{from }\kappa_{12}} \underbrace{-\frac{1}{4}(m_u+m_d)(\bar{u}u+\bar{d}d)}_{\text{from }\delta_{12}}. \end{aligned} \tag{38}$$

[2] $<\bar{d}d>$ vanishes for $m_d = 3m_u$. We shall see in subsection 7.1 that this is also the condition for the u quark to couple to the "standard" Higgs boson ξ like in the Glashow-Salam-Weinberg model.

In (38), unlike in $\mathcal{L}^{eff}_{Yukawa}$, the terms proportional to δ_{12} do not vanish because the bilinear fermion operators do not reduce to their low energy VEV's $<\bar{u}u-\bar{d}d>=\nu^3$, $<\bar{u}u+\bar{d}d>=\mu^3$. Furthermore, even if m_u is set equal to m_d, the part proportional to δ_{12}, which describes $H-K$ interplay, contributes to quark masses as much as the one proportional to δ_1 which comes from K alone. Therefore, neither the effective Lagrangian $\mathcal{L}^{eff}_{Yukawa}$ nor the "low energy truncation" of the model, that includes only one Higgs doublet, K, can correctly account for fermion masses (nor, of course, for the masses of the gauge bosons, problem which led to "technicolor" models [7]). $\mathcal{L}^{eff}_{Yukawa}$ we shall accordingly only use to deal with low energy physics of scalars and pseudoscalars, in particular to build the effective scalar potential V_{eff} in subsection 6.2.

5 Summary of the parameters

By low energy considerations, we have determined the following parameters, introduced in particular in (8) and (10):

$$\begin{aligned}
&\rho_u=\frac{3m_u-m_d}{2\sqrt{2}f_\pi},\quad \rho_d=\frac{3m_d-m_u}{2\sqrt{2}f_\pi},\quad \lambda_u=\lambda_d=\frac{g(m_u+m_d)}{4\sqrt{2}m_W},\\
&\delta_1=m_\pi^2,\quad \delta_{12}=-\kappa_{12}=\frac{gf_\pi m_\pi^2}{2m_W},\quad \delta_2=0,\\
&(m_u+m_d)<\bar{u}u+\bar{d}d>\stackrel{(20)}{=}2f_\pi^2m_\pi^2,\quad (m_u-m_d)<\bar{u}u-\bar{d}d>\stackrel{(35)}{=}-f_\pi^2m_\pi^2,\\
&v\equiv\sqrt{2}<\mathfrak{s}^0>=f_\pi,\quad \sigma\equiv\sqrt{2}<\mathfrak{s}^3>=\frac{2m_W}{g}.
\end{aligned}\tag{39}$$

These should be plugged into the renormalizable form (9) of the Yukawa Lagrangian. Note that, unlike its low energy avatar (16), it depends on κ_{12} and $\delta_{12}=-\kappa_{12}$, and not on $m_{12}^2=0$.

6 The scalar potential

6.1 A $U(2)_L\times U(2)_R$ invariant potential

We shall consider a quartic $U(2)_L\times U(2)_R$ invariant potential

$$V(K,H)=-\frac{m_H^2}{2}\,(K^\dagger K+H^\dagger H)+\frac{\lambda_H}{4}\Big((K^\dagger K)^2+(H^\dagger H)^2\Big),\tag{40}$$

which thus decomposes into two independent potentials, one for K and one for H.

This is possible because (see [1]) K and H are stable by both $SU(2)_L$ and $SU(2)_R$ and transform into each other by $U(1)_L$ and $U(1)_R$ (with the appropriate signs). This last symmetry dictates in particular the equality of the couplings (quadratic and quartic) for the two doublets.

$SU(2)_L$ breaking by $v\neq 0$ and $\sigma\neq 0$ generates three Goldstone bosons in each Higgs multiplet: $\vec{\mathfrak{p}}\in K$, the pseudoscalar singlet $\mathfrak{p}^0$ and the two charged scalars $\mathfrak{s}^\pm\in H$. This also fits the scheme according to which $v\neq 0$ and $\sigma\neq 0$ spontaneously break the chiral $U(2)_L\times U(2)_R$ down to $U(1)\times U(1)_{em}$ (see [1]); there, too, six Goldstones are generated. The pseudoscalar triplet $\vec{\mathfrak{p}}\in K$ gets a small mass from the $SU(2)_L$ invariant Yukawa couplings while the pseudoscalar singlet $\mathfrak{p}^0$ and the two charged scalars $\mathfrak{s}^\pm\in H$ must be protected from this since they are also the three Goldstones to be eaten by the $\vec{W}$ gauge bosons (see section 4). $\mathfrak{p}^0$ plays a double role in that is also the Goldstone of the breaking of $U(1)_L\times U(1)_R$ down to the diagonal $U(1)$, which, at the level of the algebra, is related to parity breaking.

$v \neq 0$ is associated with $< \bar{u}u + \bar{d}d > \neq 0$, responsible for the breaking of $SU(2)_L \times SU(2)_R$ down to $SU(2)$ with the three pions as (pseudo)-Goldstone bosons, while $\sigma \neq 0$ is associated with $< \bar{u}u - \bar{d}d > \neq 0$ which is also responsible for the breaking of the custodial $SU(2)$ into $U(1)_{em}$ and of the $\vec{W}$ mass.

Our choice for the potential amounts to requesting that, in the absence of Yukawa couplings, all fields are Goldstones but for the two Higgs bosons.

In the most general potential for two Higgs doublets the following terms have accordingly been discarded:

- $(m^2 K^\dagger H + h.c)$, with $m \in \mathbb{C}$ would mediate in particular transitions between scalars and pseudoscalars that should not occur classically;

- $\lambda_4(K^\dagger K)(K^\dagger H) + h.c.$, $\lambda_5(H^\dagger H)(K^\dagger H_2) + h.c.$ with $\lambda_4, \lambda_5 \in \mathbb{C}$ would also mediate unwanted classical transitions between scalars and pseudoscalars;

- $\lambda_3(K^\dagger H)^2 + h.c.$ with $\lambda_3 \in \mathbb{C}$ would in particular contribute to the mass of the neutral pion and not to that of the charged pions. Such a classical $\pi^+ - \pi^0$ mass difference which is not electromagnetic nor due to $m_u \neq m_d$ is unwelcome;

- $\lambda_1(K^\dagger K)(H^\dagger H)$, $\lambda_2(K^\dagger H)(H^\dagger K)$, with $\lambda_1, \lambda_2 \in \mathbb{R}$ would also spoil the Goldstone nature of the pions and η, the first because of terms proportional to $< \mathfrak{s}^3 >^2 \vec{\pi}^2$ and $< \mathfrak{s}^0 >^2 \eta^2$, the second because of terms proportional to $< \mathfrak{s}^0 >^2 \eta^2, < \mathfrak{s}^3 >^2 {\pi^0}^2$ and $< \mathfrak{s}^0 >< \mathfrak{s}^3 > \pi^0\eta$.

6.2 The low energy effective potential

At low energy, the renormalizable $V(K,H)$ is supplemented by $(-1)\times$ the bosonised form of the Yukawa Lagrangian (16). This yields the effective potential

$$\begin{aligned}V_{eff}(K,H) &= V(K,H) + \delta_1\, K^\dagger K + \frac{1}{2}m_{12}^2\,(K^\dagger H + H^\dagger K) + \delta_2\, H^\dagger H\\ &= -\frac{m_H^2}{2}\,(K^\dagger K + H^\dagger H) + \frac{\lambda_H}{4}\Big((K^\dagger K)^2 + (H^\dagger H)^2\Big) + \delta_1\, K^\dagger K + \frac{1}{2}m_{12}^2\,(K^\dagger H + H^\dagger K) + \delta_2\, H^\dagger H.\end{aligned} \tag{41}$$

It is further simplified since we have shown that $\delta_2 = 0$ and $m_{12}^2 = 0$ (see (30) and (31) in section 4) and V_{eff} accordingly reduces to

$$V_{eff}(K,H) = -\frac{m_H^2 - 2m_\pi^2}{2}\,K^\dagger K - \frac{m_H^2}{2}\,H^\dagger H + \frac{\lambda_H}{4}\Big((K^\dagger K)^2 + (H^\dagger H)^2\Big). \tag{42}$$

Last, to suitably reproduce the $\vec{\pi}$ and $\vec{W}$ masses, we know that it should have a minimum at values of v and σ given by (22) and (26). The two equations $\frac{\partial V_{eff}}{\partial \mathfrak{s}^0}\Big|_{<\mathfrak{s}^0>=\frac{f_\pi}{\sqrt{2}}} = 0$ and $\frac{\partial V_{eff}}{\partial \mathfrak{s}^3}\Big|_{<\mathfrak{s}^3>=\frac{\sqrt{2}m_W}{g}} = 0$ yield respectively $m_H^2 = \lambda_H < \mathfrak{s}^0 >^2 + 2m_\pi^2$ and $m_H^2 = \lambda_H < \mathfrak{s}^3 >^2$ such that

$$\lambda_H = \frac{2m_\pi^2}{< \mathfrak{s}^3 >^2 - < \mathfrak{s}^0 >^2} \approx \frac{2m_\pi^2}{< \mathfrak{s}^3 >^2}\left(1 + \frac{< \mathfrak{s}^0 >^2}{< \mathfrak{s}^3 >^2}\right) = \frac{g^2 m_\pi^2}{m_W^2}\left(1 + \frac{g^2 f_\pi^2}{4m_W^2}\right), \tag{43}$$

which puts it definitely in the perturbative regime. It is because of the presence of m_π^2 that λ_H is different from zero. $m_\pi \neq 0$ keeps accordingly the theory away from instability.

6.3 The masses of the two Higgs bosons ς and ξ

Since the effective scalar potential is now fully determined, one can calculate the masses of the two Higgs bosons ς and ξ defined in (11), which do not mix. One gets

$$m_\xi = < \mathfrak{s}^3 > \sqrt{\lambda_H} \approx \sqrt{2}\, m_\pi,$$

$$m_\varsigma \;=\; <\mathfrak{s}^0> \sqrt{\lambda_H} = m_\xi \frac{<\mathfrak{s}^0>}{<\mathfrak{s}^3>} \approx m_\pi \frac{g f_\pi}{\sqrt{2}\, m_W} \approx 68\, KeV, \tag{44}$$

In particular, their ratio is that of the two VEV's

$$\frac{m_\xi}{m_\varsigma} = \frac{<\mathfrak{s}^3>}{<\mathfrak{s}^0>} = \frac{2m_W/g}{f_\pi} \tag{45}$$

which is also the ratio of the two scales involved in this 1-generation standard model, the weak scale $\simeq m_W$ and the chiral scale $\simeq f_\pi$. The masses are small and justify *a posteriori* our low energy treatment of the scalar effective potential.

The composition of the two Higgs doublets is accordingly as follows. Inside the "chiral" doublet K one finds 3 pions and the very light scalar Higgs boson ς. As was shown in [1], they correspond respectively to a triplet and a singlet of the custodial $SU(2)$ symmetry. Inside the "weak" doublet H, one finds the three Goldstones of the broken $SU(2)_L$ weak symmetry, the neutral pseudoscalar $SU(2)$ singlet and two charged scalars inside the $SU(2)$ triplet. The third component of this triplet is the second scalar Higgs boson ξ with mass $\approx m_\pi$. Note that the four particles $(\vec{\pi}, \xi)$ with mass m_π do not lie together inside the same $SU(2)_L$ doublet, nor do the three $SU(2)_L$ Goldstones and the very light Higgs boson ς.

6.3.1 The roles of m_W and m_π

In our rebuilding of Standard Model with only one generation, we find that the masses of the two Higgs bosons are both proportional to m_π and small. But they are not small in the same way. If m_π is replaced by the mass of some heavier bound state $m \leq \sqrt{2} m_W/g \equiv <\mathfrak{s}^3> \approx 168\, GeV$, m_ς will stay very small $m_\varsigma \leq f_\pi \approx 93\, MeV$ while m_ξ will grow like the mass of the bound state. So, in the case of more generations, the presence of very light Higgs boson(s) with a mass lower than $100\, MeV$ looks a robust feature as a damping effect of the weak scale m_W but larger masses can be expected for some others. It would not be a surprise that, for 3 generations and up to some coefficient, the mass of one of the Higgs bosons be set by that of a bound state involving the top quark.

In the present case, the masses of the two Higgs bosons vanish at the limit $m_\pi \rightarrow 0$, that is, by the GMOR relation (20), either when $<\bar{u}u + \bar{d}d> \rightarrow 0$ or when $(m_u + m_d) \rightarrow 0$. Since we have also determined (see (39)) that $(m_u - m_d) <\bar{u}u - \bar{d}d>$ vanishes with m_π, this limit corresponds either to $<\bar{u}u> = 0 = <\bar{d}d>$ or to $m_u = 0 = m_d$.

7 Couplings of the Higgs bosons

7.1 Couplings of Higgs bosons to quarks

Like for the calculation of fermion masses (see subsection4.4), the bosonised forms (16) or (17) of the Yukawa couplings, which are only valid at low energy, is inappropriate to evaluate the couplings of fermions, in particular those to the Higgs bosons. Indeed, plugging into (16) or (17) the relations $m_{12}^2 \stackrel{(15)}{\equiv} \delta_{12} + \kappa_{12} = 0$ and $\delta_2 = 0$ that we have obtained for the crossed couplings (see (39)) from low energy considerations would erroneously leave as the only couplings of quarks to Higgs bosons the ones present in $-\delta_1 K^\dagger K$, in which, in particular, no coupling exists between the "quasi-standard" Higgs boson ξ, which belongs to H, and quarks. In order to properly determine these parameters, the original form (9) of the Yukawa couplings must instead be used.

Plugging therefore the definition (11) into (9) yields the following couplings of the Higgs bosons ς and ξ to quarks

$$\begin{aligned} & -\varsigma\,(\rho_u \bar u u + \rho_d \bar d d) - \xi\,(\lambda_u \bar u u + \lambda_d \bar d d) \\ = & -\varsigma\,\left(\delta_1 \tfrac{v}{\sqrt{2}\mu^3}(\bar u u + \bar d d) + \kappa_{12}\tfrac{\sigma}{\sqrt{2}\nu^3}(\bar u u - \bar d d)\right) - \xi\,\left(\delta_{12}\tfrac{v}{\sqrt{2}\mu^3}(\bar u u + \bar d d) + \delta_2 \tfrac{\sigma}{\sqrt{2}\nu^3}(\bar u u - \bar d d)\right) \end{aligned} \tag{46}$$

which exhibits, of course, the same structure as in (37) and which, using the values (39) of the parameters, $\delta_{12} = -\kappa_{12}$ and $\delta_2 = 0$, yields

$$\mathcal{L}_{Higgs-quarks} = -\varsigma\left(\frac{3m_u - m_d}{2\sqrt{2}f_\pi}\,\bar u u + \frac{3m_d - m_u}{2\sqrt{2}f_\pi}\,\bar d d\right) - \xi\,\frac{g(m_u + m_d)}{4\sqrt{2}m_W}\,(\bar u u + \bar d d). \tag{47}$$

The ς Higgs boson is more strongly coupled to quarks than ξ. Its coupling is still "perturbatively" since $m_u, m_d \ll f_\pi$. It however suggests that, for heavier quarks, some Higgs boson(s) could strongly couple to hadronic matter. As far as ξ is concerned, it looks at first sight "quasi-standard" because it is proportional to gm_{quark}/m_W. It is however not quite so because in the standard case we would have obtained $-\frac{g}{\sqrt{2}m_W}(m_u \bar u u + m_d \bar d d)\,\xi$. The difference is that, though u and d have different masses, they now get coupled to ξ with equal strength: unlike in the genuine Glashow-Salam-Weinberg model, the heavier quark is no more strongly coupled than the lighter. Taking $m_d = \gamma m_u, \gamma > 1$, the coupling $-\frac{g(1+\gamma)}{4\sqrt{2}m_W} m_u$ of ξ to u quarks can be very close to the standard one (it becomes identical for $\gamma = 3$, value at which $< \bar d d >$ vanishes, see footnote in subsection 4.3.2), while the one $-\frac{g(1/\gamma+1)}{4\sqrt{2}m_W} m_d$ of ξ to the heavier d is smaller than standard by the factor $\frac{(1+\gamma)}{4\gamma}$.

7.2 Couplings of Higgs bosons to gauge bosons

They arise from the kinetic terms (5). Using (22) and (26)), one gets

$$\mathcal{L}_{HiggsWW} = \frac{gm_W}{2} W_\mu^2\,\xi + \frac{g^2 f_\pi}{4\sqrt{2}} W_\mu^2\,\varsigma. \tag{48}$$

ξ couples accordingly in a "standard" way $\simeq gm_W$ to two W's while the coupling of ς, $\mathcal{O}(g^2 f_\pi)$ is much smaller by a factor $\mathcal{O}(10^{-3})$.

7.3 Couplings of Higgs bosons to leptons

Yukawa couplings to leptons need introducing four parameters, ρ_e and ρ_ν for $\mathfrak{s}^0$ and the quantum Higgs ς, λ_e and λ_ν for $\mathfrak{s}^3$ and the quantum Higgs ξ

$$\begin{aligned} \mathcal{L}_{Yuk-lept} = & \Big((\rho_\nu \bar\nu\nu + \rho_e \bar e e)\,\mathfrak{s}^0 - (\lambda_\nu \bar\nu\nu + \lambda_e \bar e e)\,\mathfrak{s}^3\Big) \\ + & \left(\frac{\rho_\nu + \rho_e}{2}\Big(\bar\nu\gamma_5 e\,\mathfrak{p}^- + \bar e\gamma_5\nu\,\mathfrak{p}^+ + (\bar\nu\gamma_5\nu - \bar e\gamma_5 e)\,\mathfrak{p}^3\Big)\right) + \frac{\rho_\nu - \rho_e}{2}\left(\Big(\bar e\nu\,\mathfrak{p}^+ - \bar\nu e\,\mathfrak{p}^- + (\bar\nu\gamma_5\nu + \bar e\gamma_5 e)\,\mathfrak{p}^3\Big)\right) \\ - & \left(\frac{\lambda_\nu + \lambda_e}{2}\Big(\bar e\gamma_5\nu\,\mathfrak{s}^+ - \bar\nu\gamma_5 e\,\mathfrak{s}^- - (\bar\nu\gamma_5\nu - \bar e\gamma_5 e)\,\mathfrak{p}^0\Big)\right) + \frac{\lambda_\nu - \lambda_e}{2}\left(\Big(\bar e\nu\,\mathfrak{s}^+ + \bar\nu e\,\mathfrak{s}^- - (\bar\nu\gamma_5\nu + \bar e\gamma_5 e)\,\mathfrak{p}^0\Big)\right). \end{aligned} \tag{49}$$

Using again (22) and (26 provides the lepton masses

$$m_e = \rho_e \frac{f_\pi}{\sqrt{2}} + \lambda_e \frac{\sqrt{2}m_W}{g}, \quad m_\nu = \rho_\nu \frac{f_\pi}{\sqrt{2}} + \lambda_\nu \frac{\sqrt{2}m_W}{g}. \tag{50}$$

7.3.1 The low energy limit

Let us use again the one-to-one correspondence between the components of the Higgs multiplets and bilinear quark operators (13). Using PCAC (19) and the Gell-Mann-Oakes-Renner relation (20), we could relate the charged pion fields $\pi^\pm$ and the charged pseudoscalar components $\mathfrak{p}^\pm$ of the Higgs doublet K by (23). Yukawa couplings (49) are then seen to trigger, among others, leptonic decays of charged pions. These come in addition to the "standard ones" obtained from the $W_\mu \partial^\mu \pi$ crossed couplings that originate from the kinetic terms (5) at low energy (see subsection 4.3.4) and which agree with PCAC usual calculations.

This means that, in a first approximation (and it is not the goal of this work to go beyond), we should take

$$\rho_\nu \approx 0 \approx \rho_e. \tag{51}$$

In case observed leptonic pion decay turn out to differ from PCAC estimates, the issue could be raised whether (51) should be revisited.

In relation with (50) the choice (51) leads to a standard coupling of the Higgs boson ξ to leptons, proportional to gm_{lepton}/m_W, while the ones of ς vanish (or are extremely close to).

8 Symmetries again

8.1 The roles of $< \bar{u}u + \bar{d}d >$ and $< \bar{u}u - \bar{d}d >$

$< \bar{u}u + \bar{d}d > \neq 0$ is the signal for what is commonly called "chiral symmetry breaking", the breaking of $SU(2)_L \times SU(2)_R$ down to the diagonal $SU(2)$. $< \bar{u}u - \bar{d}d > \neq 0$ breaks $SU(2)_L$, and the custodial $SU(2)$ down to $U(1)_{em}$. Let us show that $< \bar{u}u >$ cannot be equal to $< \bar{d}d >$. Indeed, for $\nu^3 = 0$ one gets from (10) $\delta_2 = 0 = \kappa_{12}$. Then

$$m_{12}^2 = \delta_{12} = \frac{g f_\pi m_\pi^2}{2 m_W}, \tag{52}$$

in which we used the definition of δ_{12} in (10), the GMOR relation (20) and (28).

Performing the minimization of the effective potential $V_{eff}(K,H)$ while still supposing that $V(K,H)$ is $U(2)_L \times U(2)_R$ invariant gives the two equations

$$m_H^2 = \lambda_H < \mathfrak{s}^0 >^2 + 2\delta_1 + \delta_{12}\frac{< \mathfrak{s}^3 >}{< \mathfrak{s}^0 >}, \quad m_H^2 = \lambda_H < \mathfrak{s}^3 >^2 + \delta_{12}\frac{< \mathfrak{s}^0 >}{< \mathfrak{s}^3 >}, \tag{53}$$

which yield, since $< \mathfrak{s}^3 > \gg < \mathfrak{s}^0 >$ (see (22) and (26))

$$\lambda_H \approx \frac{2\delta_1}{< \mathfrak{s}^3 >^2} + \frac{\delta_{12}}{< \mathfrak{s}^0 >< \mathfrak{s}^3 >} = \frac{3}{2}\frac{g^2 m_\pi^2}{m_W^2}. \tag{54}$$

The mass matrix of the $\mathfrak{s}^0 - \mathfrak{s}^3$ system becomes then (we use (53))

$$\left(\begin{array}{cc} \frac{\partial^2 V_{eff}}{(\partial \mathfrak{s}^0)^2} \equiv 2\lambda_H < \mathfrak{s}^0 >^2 - \delta_{12}\frac{<\mathfrak{s}^3>}{<\mathfrak{s}^0>} & \frac{1}{2}\frac{\partial^2 V_{eff}}{\partial \mathfrak{s}^0 \partial \mathfrak{s}^3} = 0 \\ \frac{1}{2}\frac{\partial^2 V_{eff}}{\partial \mathfrak{s}^0 \partial \mathfrak{s}^3} = 0 & \frac{\partial^2 V_{eff}}{(\partial \mathfrak{s}^3)^2} \equiv 2\lambda_H < \mathfrak{s}^3 >^2 - \delta_{12}\frac{<\mathfrak{s}^0>}{<\mathfrak{s}^3>} \end{array}\right) \approx \left(\begin{array}{cc} -m_\pi^2 & 0 \\ 0 & 6m_\pi^2 \end{array}\right). \tag{55}$$

It exhibits, because of the term $-\delta_{12}\frac{<\mathfrak{s}^3>}{<\mathfrak{s}^0>}$ in $\frac{\partial^2 V_{eff}}{(\partial \mathfrak{s}^0)^2}$, which comes from the low energy expression of Yukawa couplings, a tachyonic s Higgs boson $m_\varsigma^2 \approx -m_\pi^2$. The theory with $< \bar{u}u > = < \bar{d}d >$ is thus unstable.

Since we have everywhere supposed that the minimum of the effective potential fits the $\vec{W}$ and $\vec{\pi}$ masses, we conclude that chiral and weak symmetry breakings as they are observed are only possible for $< \bar{u}u > \neq < \bar{d}d >$.

Unlike for the pions the masses of which are related to $< \bar{u}u + \bar{d}d >$ by the GMOR relation (20), there is no such relation between m_W and $< \bar{u}u - \bar{d}d >$ (see the last line of (39)). Moreover, even when $< \bar{u}u > = < \bar{d}d >$

(that is, $<\nu^3>=0$) $<\mathfrak{s}^3>$ can be equal to $\sigma/\sqrt{2}$ because, in its expression (13), ν^3 cancels between the numerator and the denominator. This is why it looks opportune to rather speak of $<\bar u u>\neq<\bar d d>$ as the *catalyst* of weak (and custodial) symmetry breaking.

8.2 The custodial $SU(2)$

While $(\bar u u+\bar d d)$ gets annihilated by all generators of $SU(2)$, $(\bar u u-\bar d d)$ does not, it only gets annihilated by $T^3=Q$ (see [1]). So, $<\bar u u>\neq<\bar d d>$ spontaneously breaks the custodial $SU(2)$ down to $U(1)_{em}$. In this breaking one expects two Goldstones. They are the excitations by T^+ and T^- of the $\mathfrak{s}^3$ vacuum , that is the two scalars $\mathfrak{s}^+$ and $\mathfrak{s}^-$ eaten by $W^\pm$, and which coincide with the two charged Goldstones of the spontaneously broken weak $SU(2)_L$.

The electroweak Lagrangian is invariant by the custodial $SU(2)$ as soon as the $\vec W$'s form an $SU(2)$ vector. But, in the broken phase, the W^3 can only eat $\mathfrak{s}^0$ which is a $SU(2)$ singlet. This is how the generation of the $\vec W$ mass breaks the custodial symmetry.

8.3 Goldstones and pseudo-Goldstones

Three true Goldstones are eaten by the $\vec W$'s to get massive: they are $\mathfrak{p}^0$, $\mathfrak{s}^+$ and $\mathfrak{s}^-$, belonging to the doublet H. $\mathfrak{p}^0$ is also the Goldstone of the $U(1)_L\times U(1)_R$ spontaneous breaking down to the diagonal $U(1)$. The three $\vec{\mathfrak{p}}$ (the three pions) are the pseudo-Goldstones of the broken $SU(2)_L\times SU(2)_R$ down to $SU(2)$.

The only non-Goldstones are the two Higgs bosons ξ and ς in the sense that, though their masses also vanish with m_π, they do not seem connected with the breaking of any continuous symmetry. The first could only be excited by acting either on $\mathfrak{p}^0$ with T^3_L or T^3_R, or on $\mathfrak{p}^3$ with $\mathbb{I}_L$ or $\mathbb{I}_R$. However, in a first approximation, neither $\mathfrak{p}^0$ nor $\mathfrak{p}^3$, being a pseudoscalar, has a non-vanishing VEV. Likewise, H could only be excited either by acting on $\mathfrak{p}^3$ with T^3_L or T^3_R, or on $\mathfrak{p}^0$ with $\mathbb{I}_L$ or $\mathbb{I}_R$. The same argumentation rejects thus both as Goldstone bosons, unless some additional spontaneously broken continuous symmetry is at work, which is to be uncovered.

9 A few hints for more generations

Before concluding, it is worth pointing at a few features concerning the case of a larger number N of generations (some information can also be found in [8]). A more detailed study is postponed to [9].

There are features of this work which only belong to the case of one generation. For example the fact that the η pseudoscalar meson (pseudoscalar singlet) becomes the longitudinal neutral W^3. In the case of more generations, it may happen that this role is still held by the singlet $\propto \bar u\gamma_5 u+\bar d\gamma_5 d+\bar c\gamma_5 c+\bar d\gamma_5 s+\ldots$, but it is no longer the η, or by another neutral combination. Though this can only be known by a precise study, it is likely that the η can then live again its life as a physical pseudoscalar meson.

Other features are certainly, at the opposite, robust, like the fact that there is a very light Higgs boson with mass $\leq f_\pi\approx 93\,MeV$. Likewise, from the expression (43) for the quartic Higgs coupling λ_H, it seems reasonable to believe that, even if the mass of the pion gets replaced by the mass of a much heavier bound state, λ_H will stay smaller than 1 and thus “perturbative”. It can only get equal to 1 if m_π is replaced by $\sqrt{2}m_W/g\approx 168\,GeV$, such that one should only be careful when the “top” generation is concerned, for which “non-perturbative” phenomena could appear.

The logic of the present work and of [1] is that all (pseudo)scalar doublets isomorphic to the one of the Standard Model of Glashow, Salam and Weinberg [2] should be incorporated. This would stay an empty or meaningless statement without noticing that the standard Higgs doublet has transformations by the chiral group (3) (4) that are identical to those of bilinear quark operators. For one generation, this doubled the number of possible doublets, with parity distinguishing the two of them. In the case of N generations, it was shown in [8] that there exists $2N^2$ such doublets, divided, by parity again, in two sets. Their $8N^2$ real components can be put in one-to-one relationship with the same number of scalar and pseudoscalar $J = 0$ mesons that occur for $2N$ flavors of quarks. The same logic as the one followed here requires accordingly that the Standard Model be then endowed with $2N^2$ complex Higgs doublets. Among these, one expects in particular as many Higgs fields as there exist quark-antiquark $< \bar{q}_i q_i >$ condensates, that is, $2N$. Owing to the large number of parameters involved, it looks of course too optimistic to think that one can easily calculate all masses and couplings as we did here. This path stays nevertheless in our opinion the most natural to follow, the underlying guess being that the mystery of Higgs boson(s) simply lies inside the one of scalar (and eventually pseudoscalar) $J = 0$ mesons.

10 Conclusion and prospects

As we re-built it, the Standard Model for one generation of fermions is complete in the sense that all masses and couplings of all fields present in the Lagrangian and of all $J = 0$ pseudoscalar mesons are determined. Pions are accounted for with the correct decays and, of the four expected scalar mesons, two (the charged ones) become the longitudinal charged $W^\pm$ while the last two are the Higgs bosons ς and ξ. Both have small masses and are perturbatively coupled and self-coupled. While ξ is expected to be close to standard, ς is extremely light and has peculiar properties that deserve a specific investigation concerning the role that it can hold in nature [10]. As far as we can see, this minimal extension of the Standard Model is different from what other authors have been considering; it is different as a 2-Higgs doublet model [11] [12] [13], and it is different in that, for a larger number of generations $N > 1$, it cannot stay as a 2-Higgs doublet model and should be endowed with $2N^2$ Higgs doublets. A key ingredient to account simultaneously for the different scales in presence, weak and chiral, is parity doubling. It could only be uncovered through the one-to-one correspondence demonstrated in [1] between the Higgs fields and bilinear quark operators and detailed symmetry considerations. The breaking of parity has reflected here in the mass splitting of the two Higgs bosons, their ratio being precisely that of the two scales in presence.

At this stage, no physics "beyond the Standard Model" looks needed [3] but, since the one generation case can only be considered as a "toy Standard Model", this is one among the features that should be carefully scrutinized for more generations of fermions [9].

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[3] The only hint in favor of it may be the vanishing of the masses of the two Higgs bosons at the chiral limit, which makes them appear "like pseudo-Goldstone bosons" (see subsection 8.3).

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